PLACING N NON DOMINATING
QUEENS ON THE N x N CHESSBOARD

Part II : 19 ≤ N ≤ 45

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• Introduction

This article is the continuation of part I. It deals with larger boards.
The “productive” patterns are : **4NAQ**, **4 AQ** and **1AQ**… excepted for N = 22 for which our best placement unexpectedly uses pattern **1AQ**. In the following whenever pattern **1AQ** (or variant) is used, we show only the top left corner of the N x N board.

THREE SOLUTIONS FOR N = 19 ; U(19) = 132

S1 **1AQ**

Q	Q	Q		
Q	Q	Q	Q	
Q	Q	Q	Q	Q
	Q	Q	Q	Q
		Q	Q	Q

S2 **4AQ**

Q	Q																Q	Q
Q	Q															Q	Q	
	Q															Q	Q	
Q	Q															Q	Q	Q
Q																	Q	Q

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S3 **4NAQ**

Q		Q														Q		Q
Q		Q												Q		Q		
		Q												Q		Q		
Q		Q												Q		Q		Q
Q																Q		Q

These three placements have a first diagonal symmetry.

N.B : In the following : for N ≥ 21 only reduced size boards are shown : the relevant information is to be found in the corner(s).

ELEVEN SOLUTIONS FOR N = 20 : U(20) = 145

Q	Q	Q		Q
Q	Q	Q	Q	
Q	Q	Q	Q	Q
	Q	Q	Q	Q
		Q	Q	Q

Q	Q		Q	
Q	Q	Q		Q
Q	Q	Q	Q	
Q	Q	Q	Q	Q
	Q	Q	Q	Q

Q	Q	Q				
Q	Q	Q	Q			
Q	Q	Q	Q	Q		
	Q	Q	Q	Q		
		Q	Q	Q		Q

Q	Q	Q			
Q	Q	Q	Q		
	Q	Q	Q	Q	
Q		Q	Q	Q	Q
	Q		Q	Q	Q

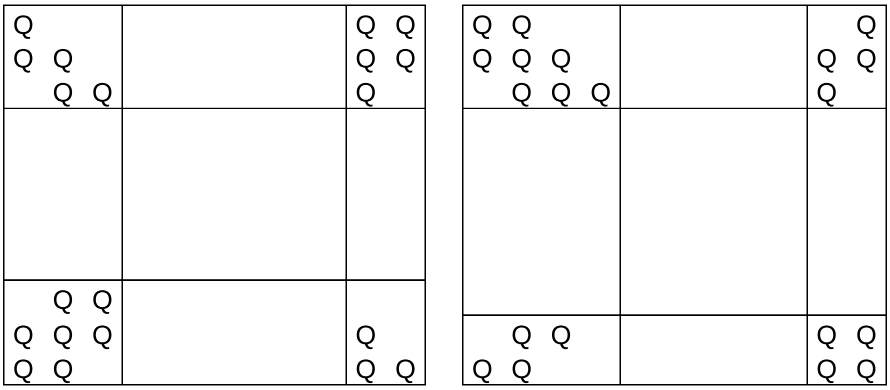
Q	Q	Q		Q	
Q	Q	Q	Q		Q
	Q	Q	Q	Q	
		Q	Q	Q	Q
			Q	Q	Q

Q	Q	Q			
Q	Q	Q	Q		
Q	Q	Q	Q		
Q	Q	Q	Q		Q
	Q	Q	Q		Q

Q	Q	Q	Q		
	Q	Q	Q	Q	
Q		Q	Q	Q	Q
	Q		Q	Q	Q
		Q		Q	Q

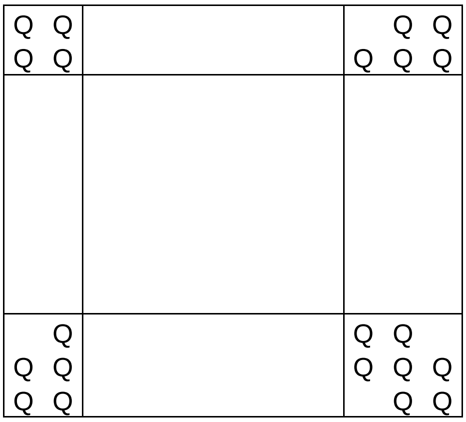
Q	Q	Q	Q		
Q	Q	Q	Q		
Q	Q	Q	Q		Q
	Q	Q	Q		Q
		Q	Q		Q

Q	Q		Q		
Q	Q	Q		Q	
Q	Q	Q	Q		Q
	Q	Q	Q	Q	
		Q	Q	Q	Q

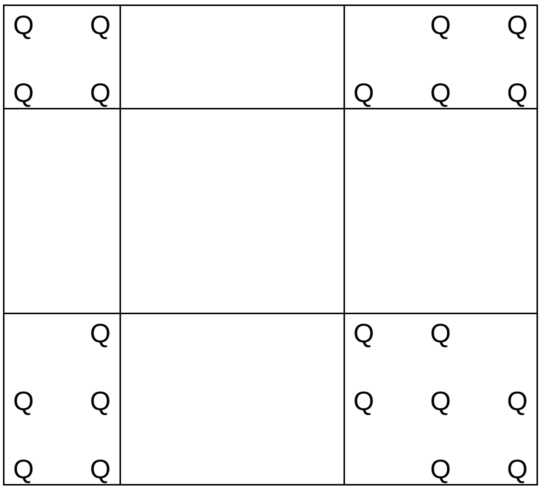


TWO SOLUTIONS FOR N = 21 : U(21) = 170

4AQ



4NAQ



A SOLUTION FOR N = 22 : U(22) = 186

To left corner

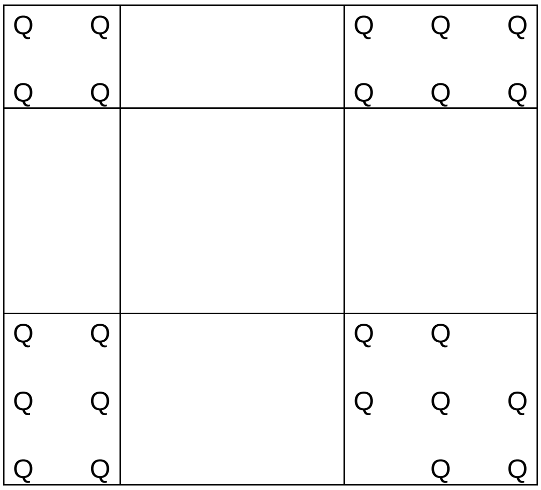
Q		Q		Q		Q			
Q		Q		Q		Q		Q	
Q		Q		Q		Q		Q	
Q		Q		Q		Q		Q	
		Q		Q		Q		X	

First diagonal symmetry

Pattern : **1NAQ** ; this pattern has provided so far best solutions only for N = 12 and N = 22.

If a 23 rd queen is added on the cross and the pattern is placed on the 23 x 23 board, it provides only U = 210 < 216 = U(23).

A SOLUTION FOR N = 23 : U(23) = 216

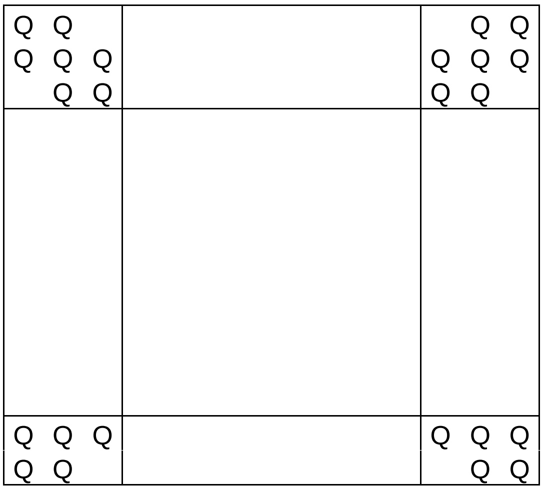


First diagonal symmetry

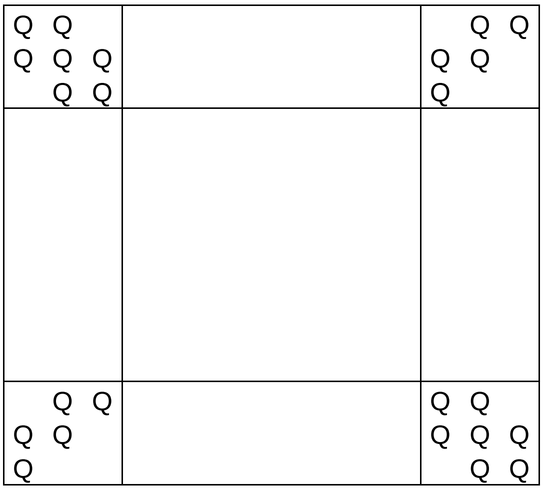
This case was a hard nut to crack ! We conjecture that there exists no solution with :
211 < U < 216.

FOUR SOLUTIONS FOR N = 24 : U(24) = 240

S1 **4AQ**



S2 **4AQ**



S3 **1AQ**

Q	Q	Q		
Q	Q	Q	Q	
Q	Q	Q	Q	Q
Q	Q	Q	Q	Q
	Q	Q	Q	Q
		Q	Q	Q

S4 **1AQ**

Q	Q	Q			
Q	Q	Q	Q		
Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q
		Q	Q	Q	Q
			Q	Q	Q

Placements S1 has a vertical axial symmetry. S2 and S4 have a first diagonal symmetry

THREE SOLUTIONS FOR N = 25 : U(25) = 260

4NAQ

Q	Q			Q	Q
Q	Q		Q	Q	Q
Q	Q		Q	Q	
	Q		Q		
Q	Q		Q	Q	Q
Q	Q		Q	Q	Q

4NAQ

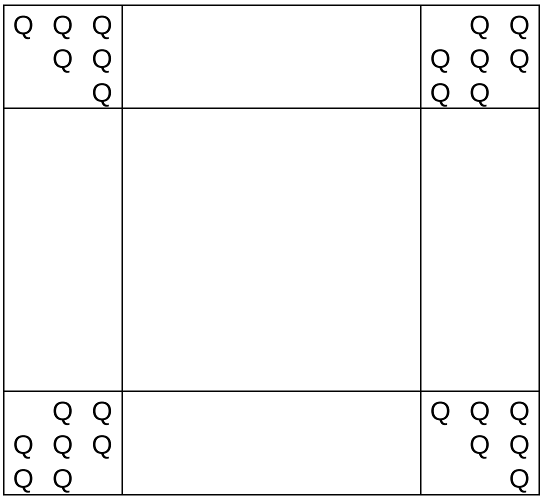
Q	Q			Q	Q
Q	Q		Q	Q	Q
Q	Q		Q	Q	
			Q	Q	
Q	Q		Q	Q	Q
Q	Q		Q	Q	Q

4NAQ

Q	Q			Q	Q
Q	Q		Q	Q	Q
Q	Q			Q	
	Q		Q	Q	
Q	Q		Q	Q	Q
Q	Q		Q	Q	Q

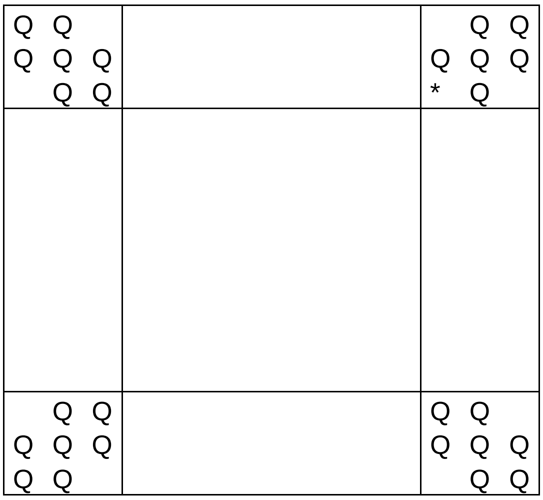
These 3 placements have no symmetry.

TWO SOLUTIONS FOR N = 26 : U(26) = 290



Q	Q		Q		
Q	Q	Q		Q	
Q	Q	Q	Q		Q
Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q
		Q	Q	Q	Q

TWO SOLUTIONS FOR N = 27 : U (27) = 324



Q	Q	Q	Q		
Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q
		Q	Q	Q	Q
			Q	Q	Q

TWO SOLUTIONS FOR N = 28 : U(28) = 360

Q Q Q Q Q Q Q		Q Q Q Q Q Q Q
Q Q Q Q Q Q Q		Q Q Q Q Q Q Q

Q Q Q Q Q Q Q		Q Q Q Q Q Q Q
Q Q Q Q Q Q Q		Q Q Q Q Q Q Q

SIXTEEN SOLUTIONS FOR N = 29 : U(29) = 381

4AQ

Q Q Q Q Q Q Q		Q Q Q Q Q Q Q
Q Q Q Q Q Q Q Q		Q Q Q Q Q Q Q Q

Vertical axial symmetry

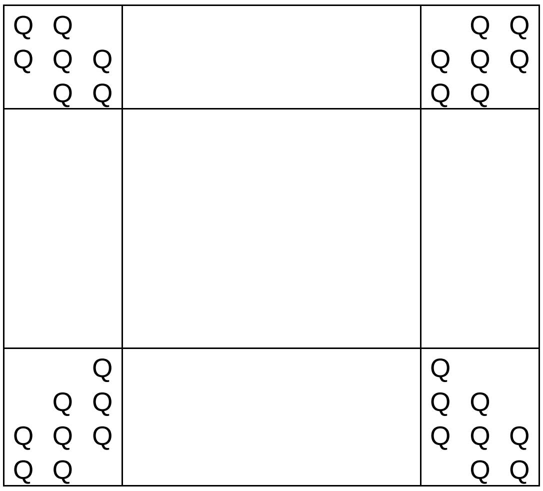
1AQ

Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q
		Q	Q	Q	Q
			Q	Q	Q

On the 29 x 29 board above, 30 queens are placed leaving 420 unattacked square. Removing any one of these queens provides a solution for N = 29 with U(29) = 381. Due to the symmetry of the above placement, one can obtain this way : 30/2 = 15 different solutions. Adding the 1AQ pattern solution, we obtain altogether 16 solutions.

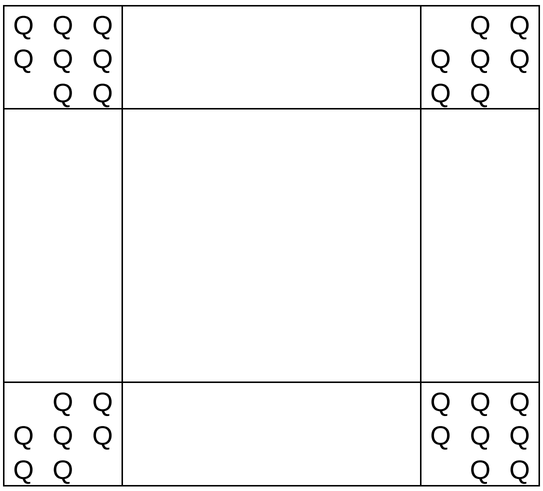
THREE SOLUTIONS FOR N = 30 : U(30) = 420

4AQ



Vertical axial symmetry

4AQ



Vertical axial symmetry

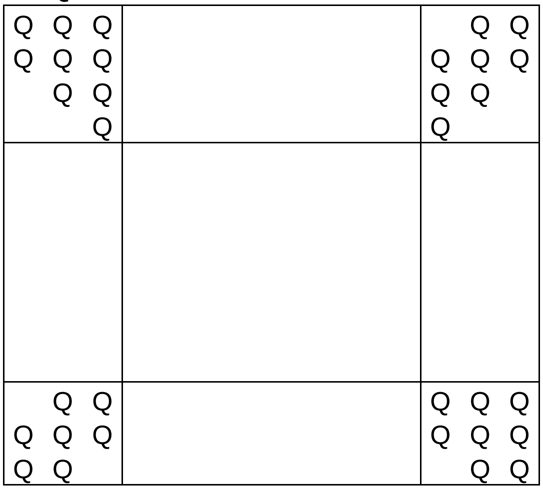
1AQ

Q	Q	Q	Q		Q
Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q
		Q	Q	Q	Q

First diagonal symmetry

THIRTY SIX SOLUTIONS FOR N = 31 : U(31) = 442

4AQ



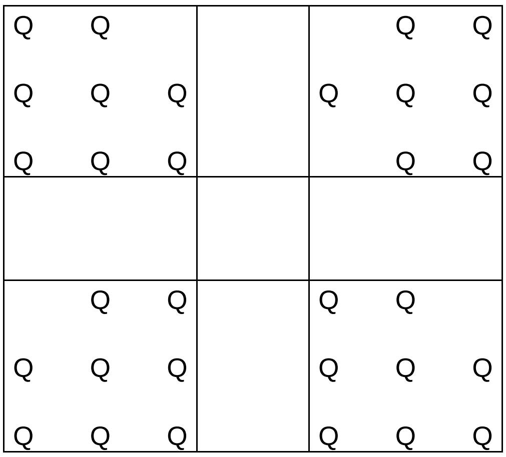
1AQ : variant

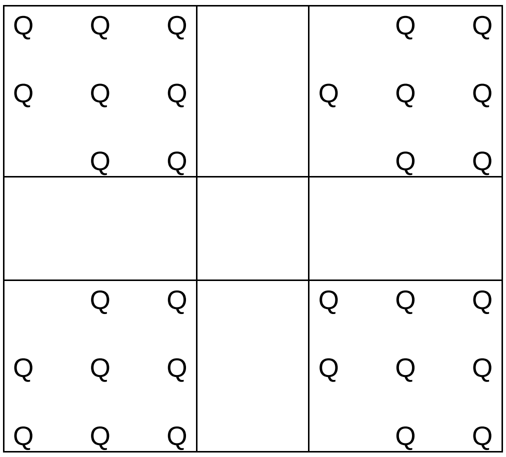
Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q
Q		Q	Q	Q	Q
	Q		Q	Q	Q

1AQ : variant

Q	Q	Q	Q		Q
Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q
		Q	Q	Q	Q

To the left above 32 queens are placed on the 31 x 31 board, leaving 442 = U(31) unattacked cases. By removing any one of these 32 queens, a solution is obtained. Since the above placement has no symmetry, 32 different solutions can be obtained. Altogether : 32 + 2 = 34 solutions. Also Johan Claes recently sent us the two basic solutions below, with pattern 4NAQ :





A SOLUTION FOR N = 32

U(32) = 485

4AQ

Q Q Q Q Q Q Q Q Q		Q Q Q Q Q Q Q Q
Q Q Q Q Q Q Q		Q Q Q Q Q Q Q Q

Vertical axial symmetry

There exists many suboptimal solutions with U = 484.

A SOLUTION FOR N = 33

U(33) = 530

4AQ

Q Q Q Q Q Q Q Q Q Q		Q Q Q Q Q Q
Q Q Q Q Q Q		Q Q Q Q Q Q Q

First diagonal symmetry

TWO SOLUTIONS FOR N = 34

U(34) = 554

Q	Q	Q	Q			
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
Q		Q	Q	Q	Q	Q

Q	Q	Q	Q			Q
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q

Variants of **1AQ**

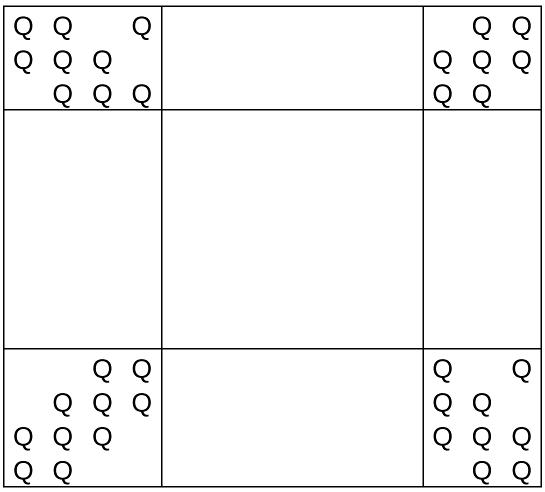
SIX SOLUTIONS FOR N = 35 ; U(35) = 602

Q	Q	Q	Q			
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q
Q			Q	Q	Q	Q
	Q			Q	Q	Q

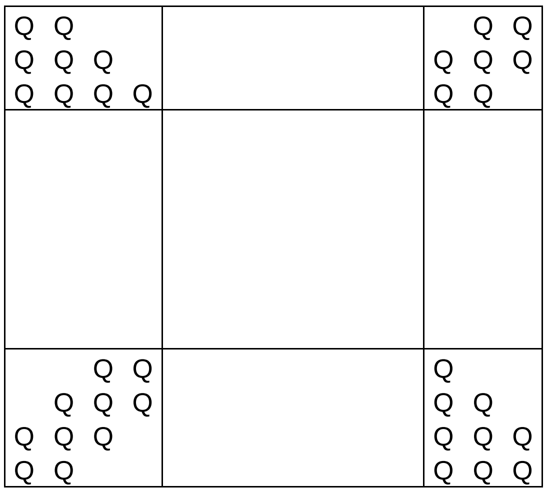
Q	Q	Q	Q		Q	
Q	Q	Q	Q	Q		Q
Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q
			Q	Q	Q	Q
				Q	Q	Q

Q	Q	Q	Q					
Q	Q	Q	Q	Q				
Q	Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q			
	Q	Q	Q	Q	Q			
		Q	Q	Q	Q			Q
			Q	Q	Q			Q

Q	Q	Q	Q				
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q		Q	
	Q	Q	Q	Q		Q	Q
		Q	Q	Q		Q	Q
			Q	Q		Q	Q



Second diagonal



Second diagonal symmetry

THREE SOLUTIONS FOR N = 36

U(36) = 650

4AQ

Q Q Q Q Q Q Q Q Q Q		Q Q Q Q Q Q Q Q
Q Q Q Q Q Q Q Q Q Q		Q Q Q Q Q Q Q Q

Horizontal axial symmetry

1AQ

Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q

First diagonal symmetry

1AQ

Q	Q	Q	Q				
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
	Q	Q	Q	Q	Q	Q	
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q
				Q	Q	Q	Q

No symmetry

A SOLUTION FOR N = 37

U(37) = 702

1AQ

Q	Q	Q	Q			
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q
			Q	Q	Q	Q

First diagonal symmetry

TWO SOLUTIONS FOR N = 38

U(38) = 731

Variant **1AQ**

Q	Q	Q	Q			
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	
Q			Q	Q	Q	Q

Variant **1AQ**

Q	Q	Q	Q			Q	
Q				Q			Q
Q					Q		
	Q					Q	
		Q					Q
			Q				Q
				Q	Q	Q	Q

TWO SOLUTIONS FOR N = 39

U(39) = 785

Q	Q	Q	Q			
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
Q		Q	Q	Q	Q	Q
	Q		Q	Q	Q	Q

Q	Q	Q	Q		Q		
Q	Q	Q	Q	Q		Q	
Q	Q	Q	Q	Q	Q		Q
	Q	Q	Q	Q	Q	Q	
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q
				Q	Q	Q	Q

Variant of **1AQ**

FOUR SOLUTIONS FOR N = 40

U(40) = 841

4AQ

Q	Q	Q				Q	Q
Q	Q	Q	Q		Q	Q	Q
	Q	Q	Q		Q	Q	
		Q	Q		Q		
		Q	Q		Q		
	Q	Q	Q		Q	Q	Q
Q	Q	Q			Q	Q	Q
Q	Q					Q	Q

Horizontal axial symmetry

1AQ

Q	Q	Q	Q			
Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q
Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q

1AQ

Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q
				Q	Q	Q	Q

1AQ

Q	Q	Q	Q				
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q

A SOLUTION FOR N = 41

U(41) = 873

Q	Q	Q	Q				Q
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q

A SOLUTION FOR N = 42

U(42) = 932

Q	Q	Q	Q			Q	
Q	Q	Q	Q	Q			Q
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q

TWO SOLUTIONS FOR N = 43

U(43) = 993

Q	Q	Q	Q				
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q

Q	Q	Q	Q		Q		
Q	Q	Q	Q	Q		Q	
Q	Q	Q	Q	Q	Q		Q
Q	Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q

TWO SOLUTIONS FOR N = 44

U(44) = 1056

Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q

Q	Q	Q	Q				
Q	Q	Q	Q	Q			
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q
				Q	Q	Q	Q

A SOLUTION FOR N= 45

U(45) = 1091

Q	Q	Q	Q	Q			Q
Q	Q	Q	Q	Q	Q		
Q	Q	Q	Q	Q	Q	Q	
Q	Q	Q	Q	Q	Q	Q	Q
	Q	Q	Q	Q	Q	Q	Q
		Q	Q	Q	Q	Q	Q
			Q	Q	Q	Q	Q

TABLE of U(N) : 4 ≤ N ≤ 45

N	4	5	6	7	8	9	10	11	12	13	14	15	16	17
U (N)	1	3	5	7	11	18	22	30	36	47	56	72	82	97
Nb sol	25	1	3	38	7	1	1	2	7	1	4	3	1	1

N	18	19	20	21	22	23	24	25	26	27	28	29	30	31
U (N)	111	132	145	170	186	216	240	260	290	324	360	381	420	442
Nb sol	2	3	1	2	1	1	4	3	2	2	2	16	3	34

N	32	33	34	35	36	37	38	39	40	41	42	43	44	45
U (N)	485	530	554	603	650	702	731	785	841	873	932	993	1056	1091
Nb sol	1	1	2	1	3	1	2	2	4	1	1	2	2	1

CONCLUSION

The problem is still open : prove wether the placements for N ≥17 are optimal or not …

Also for each optimal value of U(N), find the number of different “basic” placements of N queens leaving U(N) unattacked squares.

Also for N tending to infinity, which is (are) the best types (patterns)of placements …

Thanks to Johan Claes for providing us many new placements of his own.